Calculus 1
Final Exam – Solutions
November 4, 2022 (8:30 – 10:30)

university of groningen

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1) Prove using the ε-δ definition that $\lim_{x\to 1} \frac{3x^2-3}{x-1} = 6$.

Solution. Let $\varepsilon > 0$ be arbitrary and take $\delta = \frac{\varepsilon}{3}$. Then, $0 < |x-1| < \delta$ implies that

$$\left|\frac{3x^2-3}{x-1} - 6\right| = \left|\frac{3(x-1)(x+1)}{x-1} - 6\right| = |3(x+1)-6| = 3\,|x-1| < 3\delta = \varepsilon.$$

Thus, $\lim_{x\to 1} \frac{3x^2-3}{x-1} = 6$.

2) Apply l'Hospital's Rule to find the following limit: $\lim_{h\to 0} \frac{(1+h)^{1/h}-e}{h}$.

Solution. This limit is an indeterminate form of type $\frac{0}{0}$ since $\lim_{h\to 0}(1+h)^{1/h} = e$. To apply l'Hospital's Rule, we need to compute the derivative of the function $(1+h)^{1/h}$ which is of the form $f(h)^{g(h)}$ suggesting that logarithmic differentiation may be of use. Indeed, we have

$$\ln\left[(1+h)^{1/h}\right] = \frac{1}{h}\ln(1+h)$$

(by a law of Logarithms) and differentiating both sides yields

$$\frac{[(1+h)^{1/h}]'}{(1+h)^{1/h}} = \frac{\frac{h}{1+h} - \ln(1+h)}{h^2}. \tag{1}$$

On the left-hand side, we used the Chain Rule and on the right-hand side we employed the Quotient Rule for $\frac{\ln(1+h)}{h}$, the Sum Rule for $1+h$ and the Basic Derivative $(h)' = 1$. We also used the derivative of $\ln$ on both sides. Multiplying both sides of eq. (1) by $(1+h)^{1/h}$ yields

$$[(1+h)^{1/h}]' = \frac{\frac{h}{1+h} - \ln(1+h)}{h^2}(1+h)^{1/h}$$

and l'Hospital's Rule turns the limit in question into the following

$$\lim_{h\to 0} \frac{(1+h)^{1/h}-e}{h} = \lim_{h\to 0} \frac{[(1+h)^{1/h}-e]'}{[h]'} = \lim_{h\to 0} \frac{\frac{\frac{h}{1+h} - \ln(1+h)}{h^2}(1+h)^{1/h} - 0}{1}. \tag{2}$$

We used the Difference Rule and the Basic Derivatives $(\text{constant})' = 0$, $(h)' = 1$. Again, recall that $\lim_{h\to 0}(1+h)^{1/h} = e$, therefore finding the original limit is, due to the Product Law of Limits, equivalent to finding

$$\lim_{h\to 0} \frac{\frac{h}{1+h} - \ln(1+h)}{h^2}. \tag{3}$$

This limit is an indeterminate form of type $\frac{0}{0}$ so we may use l'Hospital's Rule to get

$$\lim_{h\to 0}\frac{\frac{h}{1+h}-\ln(1+h)}{h^2}=\lim_{h\to 0}\frac{\left[\frac{h}{1+h}-\ln(1+h)\right]'}{[h^2]'}=\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-\frac{1}{1+h}}{2h}.$$

Here we used the Quotient Rule for $\frac{h}{1+h}$, the derivative of $\ln$ as well as the Sum, Difference and Chain Rules. Combining terms in the numerator of the last expression lets us evaluate limit (3) by Direct Substitution:

$$\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-\frac{1}{1+h}}{2h}=\lim_{h\to 0}\frac{-\frac{\cancel{h}}{(1+h)^2}}{2\cancel{h}}=-\frac{1}{2}\lim_{h\to 0}\frac{1}{(1+h)^2}=-\frac{1}{2}\frac{1}{(1+0)^2}=-\frac{1}{2}.$$

This limit together with eq. (2) shows that

$$\lim_{h\to 0}\frac{(1+h)^{1/h}-e}{h}=\lim_{h\to 0}\frac{\frac{h}{1+h}-\ln(1+h)}{h^2}\lim_{h\to 0}(1+h)^{1/h}=-\frac{1}{2}\cdot e=-\frac{e}{2}.$$

3) We say that a function f has a *fixed point* at $a\in\operatorname{dom}(f)$ if $f(a)=a$. Prove the following statement. [Hint: Use the Mean Value Theorem.]

> "If $f\colon\mathbb{R}\to\mathbb{R}$ is differentiable everywhere and $f'(x)\neq 1$ for all $x\in\mathbb{R}$, then f has at most one fixed point."

Solution. We will prove the statement by contradiction.

Proof. Assume f has at least two *distinct* fixed points, say at a and b ($a\neq b$). Without loss of generality, we may assume that $a<b$. Then, since the function is differentiable everywhere, it is differentiable on the closed interval $[a,b]$. This implies that f continuous on $[a,b]$, and also differentiable on (a,b). Then, by the Mean Value Theorem, there is a $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

But we also have $f(a)=a$ and $f(b)=b$ since a and b are fixed points. Thus we get

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{b-a}{b-a}=1.$$

This contradicts the condition that $f'(x)\neq 1$ at every $x\in\mathbb{R}$. Therefore we conclude that our initial assumption of f having at least two distinct fixed points must have been false, i.e. f has at most one fixed point. $\square$

(Any other correct attempt also gets full marks).

4) Use the Maclaurin series for $f(x)=(1+x^2)\ln(1+x)$ to find $f^{(2022)}(0)$.

Solution. On the one hand, $f^{(2022)}(0)$ appears in the coefficient c_{2022} of x^{2022} in the Maclaurin series for f. Namely, we have

$$c_{2022}=\frac{f^{(2022)}(0)}{2022!} \tag{4}$$

[see Theorem 5 in Section 11.10 of the Textbook]. On the other hand, the same coefficient of x^{2022} can be found by multiplying the Maclaurin series for $\ln(1+x)$ [Table 1 in Section 11.10] by x^2 to get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots - \frac{x^{2020}}{2020} + \frac{x^{2021}}{2021} \boxed{-\frac{x^{2022}}{2022}} + \ldots$$

$$x^2\ln(1+x) = x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \ldots \boxed{-\frac{x^{2022}}{2020}} + \frac{x^{2023}}{2021} - \frac{x^{2024}}{2022} + \ldots$$

and then adding the two series [see "Taylor Series from Old" in Section 11.10] to find that the coefficient of x^{2022} is

$$c_{2022} = -\frac{1}{2022} - \frac{1}{2020} = -\frac{4042}{2022 \cdot 2020}. \tag{5}$$

Comparing eqs. (4) and (5), we obtain

$$f^{(2022)}(0) = -\frac{4042}{2022 \cdot 2020} 2022! = -2 \cdot 2021^2 \cdot 2019!.$$

5) Prove, via mathematical induction on N, that

$$\int_0^\infty x^N e^{-x}\, dx = N! \quad \text{for every integer } N \geq 0.$$

(Recall that $0! = 1! = 1$ and $n! = n \cdot (n-1) \cdots 2 \cdot 1$ for $n = 2, 3, \ldots$)

Solution. As told, we use proof by induction.

Proof. **Base Case**: For $N = 0$, we find

$$\int_0^\infty x^0 e^{-x}\, dx = \int_0^\infty e^{-x}\, dx = \lim_{b\to\infty} \int_0^b e^{-x}\, dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} (1 - e^{-b}) = 1 = 0!$$

Above we used the definition of improper integrals, the Difference Law and $\lim_{b\to\infty} e^{-b} = 0$.

Induction Hypothesis: Next, we assume that the statement holds for *some* integer $n \geq 0$, i.e. that we have

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

Induction Step: Finally, we prove, using the induction hypothesis, that the statement also holds $n+1$. We start with the integral on the left-hand side and apply integration by parts:

$$\begin{aligned}\int_0^\infty x^{n+1} e^{-x}\, dx &= \lim_{b\to\infty} \int_0^b x^{n+1} e^{-x}\, dx \overset{(*)}{=} \lim_{b\to\infty} \left[-x^{n+1}e^{-x}\right]_0^b - \lim_{b\to\infty} \int_0^b (n+1)x^n \cdot -e^{-x}\, dx \\ &= \lim_{b\to\infty} (-b^{n+1}e^{-b}) + (n+1) \int_0^\infty x^n e^{-x}\, dx \overset{(**)}{=} (n+1) \cdot n! = (n+1)!\end{aligned}$$

At $(*)$, we integrate e^{-x} and differentiate x^{n+1} and use the Difference Law of Limits. At $(**)$ we use the induction hypothesis and the limit $\lim_{b\to\infty} (-b^{n+1}e^{-b}) = 0$ that can be shown via l'Hospital's Rule (applied $(n+1)$-times). Thus the induction step is done and the proof is complete. □

To conclude, we have shown by mathematical induction on N, that

$$\int_0^\infty x^N e^{-x}\,dx = N! \quad \text{for every integer } N \geq 0.$$

6) Determine the surface area of the solid obtained by rotating the curve $C = \{(x, \cosh x) \mid 0 \leq x \leq 1\}$ about the x-axis. Include all sides!

Solution. The surface in question can be divided up into three pieces. S_1: $y = \cosh x$ rotated about the x-axis, S_2: the disk on the left-hand side at $x = 0$, and S_3: the disk on the right-hand side at $x = 1$. First, we calculate the area of $y = \cosh x$ rotated about the x-axis using $2\pi \int_a^b y(x)\sqrt{1 + (dy/dx)^2}\,dx$ [see the lecture notes or Equation 4 in Section 8.2]. We find

$$\begin{aligned} S_1 &= 2\pi \int_0^1 \cosh x \sqrt{1 + \left(\frac{d\cosh x}{dx}\right)^2}\,dx = 2\pi \int_0^1 \cosh x \sqrt{1 + (\sinh x)^2}\,dx \\ &= 2\pi \int_0^1 \cosh x \sqrt{(\cosh x)^2}\,dx = 2\pi \int_0^1 (\cosh x)^2\,dx. \end{aligned}$$

There are multiple ways to evaluate this integral. One possibility is to use the (exponential) definition of the hyperbolic cosine. Recall $\cosh x = \frac{e^x + e^{-x}}{2}$. Hence $(\cosh x)^2 = (\frac{e^x+e^{-x}}{2})^2 = \frac{e^{2x}+2+e^{-2x}}{4}$. Substituting this in the integral yields

$$\begin{aligned} S_1 &= 2\pi \int_0^1 (\cosh x)^2\,dx = 2\pi \int_0^1 \frac{e^{2x} + 2 + e^{-2x}}{4}\,dx \\ &= \frac{\pi}{2}\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^1 = \pi\left(\frac{e^2}{4} + 1 - \frac{e^{-2}}{4}\right). \end{aligned}$$

Next, we calculate the surface areas of the circles that 'cap off' the surface rotated about the x-axis. These circles have radii $\cosh 0 = 1$ and $\cosh 1 = \frac{e^1+e^{-1}}{2}$, respectively. Thus we get $S_2 = \pi$ and $S_3 = \pi(\cosh 1)^2 = \pi(\frac{e^2}{4} + \frac{1}{2} + \frac{e^{-2}}{4})$. Now the total surface area is given by the sum of S_1, S_2 and S_3:

$$S = S_1 + S_2 + S_3 = \pi\left(\frac{e^2}{4} + 1 - \frac{e^{-2}}{4} + 1 + \frac{e^2}{4} + \frac{1}{2} + \frac{e^{-2}}{4}\right) = \pi\frac{5 + e^2}{2}.$$

Note: if your answer looked nothing like this, fear not.

Another way to solve the problem is by evaluating the integral S_1 and the radius in S_3 in terms of hyperbolic functions via the hyperbolic identities $\cosh^2 x - \sinh^2 x = 1$ and $\cosh 2x = \cosh^2 x + \sinh^2 x$. This way we obtain

$$\begin{aligned} S_1 &= 2\pi \int_0^1 \cosh^2 x\,dx = 2\pi \int_0^1 \frac{1 + \cosh 2x}{2}\,dx = \pi \int_0^1 (1 + \cosh 2x)\,dx = \pi\left[x + \frac{\sinh 2x}{2}\right]_0^1 \\ &= \pi\left(1 + \frac{\sinh 2}{2}\right) \end{aligned}$$

and

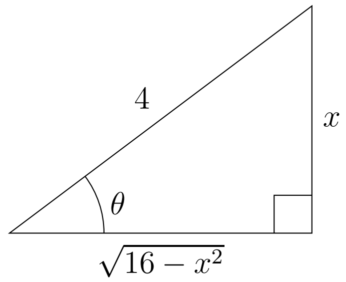
$$S_3 = \pi \cosh^2 1 = \pi\frac{1 + \cosh 2}{2}.$$

Thus the total surface area can be written as

$$S = S_1 + S_2 + S_3 = \pi\left(1 + \frac{\sinh 2}{2} + 1 + \frac{1}{2} + \frac{\cosh 2}{2}\right) = \pi\frac{5 + \cosh 2 + \sinh 2}{2} = \frac{5 + e^2}{2}.$$

7) Evaluate the definite integral $\displaystyle\int_2^{2\sqrt{3}} \frac{x+1}{x\sqrt{16-x^2}}\,dx.$

Solution. The factor $\sqrt{16-x^2}$ in the integrand suggests that a trigonometric substitution should be used. Such a square root appears as the length of one of the legs in a right triangle with the other leg being of length x and the hypotenuse of length 4.



If θ denotes the angle opposite the leg of length x, then we can express x using θ by simply using the (right triangle) definition of the sine function:

$$\sin\theta = \frac{x}{4} \quad \Rightarrow \quad \boxed{x = 4\sin\theta} \quad \Rightarrow \quad dx = 4\cos\theta\,d\theta.$$

Of course, we can also express the length of the other leg in terms of θ as $\sqrt{16-x^2} = 4\cos\theta$. Finally, we look at the limits of integration and see that $x = 2$ means $\sin\theta = \frac{1}{2}$, i.e. $\theta = \frac{\pi}{6}$ and $x = 2\sqrt{3}$ implies $\sin\theta = \frac{\sqrt{3}}{2}$, i.e. $\theta = \frac{\pi}{3}$. Thus the substitution $x = 4\sin\theta$ results in the following

$$\int_2^{2\sqrt{3}} \frac{x+1}{x\sqrt{16-x^2}}\,dx = \int_{\pi/6}^{\pi/3} \frac{4\sin\theta + 1}{4\sin\theta\cancel{4\cos\theta}}\cancel{4\cos\theta}\,d\theta = \int_{\pi/6}^{\pi/3}\left(1 + \frac{1}{4\sin\theta}\right)d\theta.$$

Integrating term by term, we see that the first term is a basic integral that yields

$$\int_{\pi/6}^{\pi/3} 1\,d\theta = [\theta]_{\pi/6}^{\pi/3} = \left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{6},$$

whereas the second term is a trigonometric integral that can be evaluated via the Weierstrass substitution $u = \tan(\theta/2)$, which recall means that $\sin\theta = 2u/(1+u^2)$ and $d\theta = 2\,du/(1+u^2)$. As for the limits of integration, we have $\theta = \pi/6$ giving $u = \tan(\pi/12)$ and $\theta = \pi/3$ giving $u = \tan(\pi/6)$. One may express these values more explicitly using trigonometric identities, although it's not strictly necessary. We get

$$\int_{\pi/6}^{\pi/3} \frac{1}{4\sin\theta}\,d\theta = \int_{\tan(\pi/12)}^{\tan(\pi/6)} \frac{\cancel{1+u^2}}{4\cdot\cancel{2}u}\,\frac{\cancel{2}}{\cancel{1+u^2}}\,du = \int_{\tan(\pi/12)}^{\tan(\pi/6)} \frac{1}{4u}\,du = \left[\frac{\ln u}{4}\right]_{\tan(\pi/12)}^{\tan(\pi/6)}.$$

Therefore the integral in question evaluates to

$$\int_2^{2\sqrt{3}} \frac{x+1}{x\sqrt{16-x^2}}\,dx = \frac{\pi}{6} + \frac{\ln\tan(\pi/6) - \ln\tan(\pi/12)}{4}.$$

Remark. Using the Laws of Logarithms we may write the result as follows

$$\frac{\pi}{6} + \frac{1}{4} \ln \frac{\tan(\pi/6)}{\tan(\pi/12)}.$$

Of course, it is possible to express the tangent values more explicitly using trigonometric identities. Namely, we have $\tan(\pi/6) = \frac{\sin(\pi/6)}{\cos(\pi/6)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$ and

$$\tan(\pi/12) = \frac{\sin(\pi/12)}{\cos(\pi/12)} = \sqrt{\frac{\sin^2(\pi/12)}{\cos^2(\pi/12)}} = \sqrt{\frac{1-\cos(2\pi/12)}{1+\cos(2\pi/12)}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}} = \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$$
$$= \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}}} = \sqrt{\frac{(2-\sqrt{3})^2}{2^2-\sqrt{3}^2}} = 2-\sqrt{3}.$$

This means that

$$\frac{\tan(\pi/6)}{\tan(\pi/12)} = \frac{1}{2\sqrt{3}-3} = \frac{2\sqrt{3}+3}{(2\sqrt{3})^2-3^2} = \frac{2}{\sqrt{3}}+1$$

hence the integral evaluates to

$$\frac{\pi}{6} + \frac{1}{4} \ln\left(\frac{2}{\sqrt{3}}+1\right) = 0.71551\ldots.$$

8) Solve the initial value problem $y'(x) - \dfrac{y(x)}{x} = x^2 + 3x - 2$, $y(1) = 4$.

Solution. This is a first-order linear ODE, thus we can use an integrating factor to solve it. Namely, we have $I(x) = e^{\int\left(-\frac{1}{x}\right)dx} = e^{-\ln x} = \frac{1}{x}$ and multiplying both sides of the equation by it yields

$$\frac{y'(x)}{x} - \frac{y(x)}{x^2} = x + 3 - \frac{2}{x}.$$

Now we have the left-hand side equal to $\left(\dfrac{y(x)}{x}\right)'$, so we have

$$\left(\frac{y(x)}{x}\right)' = x + 3 - \frac{2}{x}.$$

Integrating on both sides with respect to x yields

$$\frac{y(x)}{x} = \frac{x^2}{2} + 3x - 2\ln(x) + C$$

from which we get the general solution

$$y(x) = \frac{x^3}{2} + 3x^2 - 2x\ln(x) + Cx.$$

This combined with the initial condition $y(1) = 4$ means that

$$4 = y(1) = \frac{1}{2} + 3 - 0 + C \quad \Rightarrow \quad C = \frac{1}{2}.$$

Therefore the solution for the given initial value problem is

$$y(x) = \frac{x^3}{2} + 3x^2 - 2x\ln(x) + \frac{x}{2}, \qquad (x > 0).$$

9) Find all complex number solutions of the equation $z^2 = \overline{z}$. Write your final answer in algebraic form.

Solution. Writing $z = a + ib$ allows us to express the left-hand side of the equation as

$$z^2 = (a + ib)^2 = a^2 + i2ab - b^2.$$

This gives the real part $\mathrm{Re}(z^2) = a^2 - b^2$ and the imaginary part $\mathrm{Im}(z^2) = 2ab$. On the other hand, we can also express the right-hand side as:

$$\overline{z} = a - ib$$

which gives the real part $\mathrm{Re}(\overline{z}) = a$ and the imaginary part $\mathrm{Im}(\overline{z}) = -b$. To have $z^2 = \overline{z}$ we need the real parts to match $\mathrm{Re}(z^2) = \mathrm{Re}(\overline{z})$ and also the imaginary parts to match $\mathrm{Im}(z^2) = \mathrm{Im}(\overline{z})$. Thus we get the following system of equations for a and b:

$$\begin{cases} a^2 - b^2 = a \\ 2ab = -b \end{cases} \quad \Leftrightarrow \quad \begin{cases} a(a-1) = b^2 \\ (2a+1)b = 0 \end{cases}$$

From the second equation we see that we must have $a = -\frac{1}{2}$ or $b = 0$. In fact, we can't have both, because that would lead to the contradiction $\frac{3}{4} = 0$ in the first equation. Thus we have two distinct cases to consider:

- Case 1: If $b = 0$, then the first equation reads $a(a-1) = 0$ implying $a = 0$ or $a = 1$. This gives the two solutions of the original equation $z^2 = \overline{z}$, namely $z = 0$ and $z = 1$.
- Case 2: If $a = -\frac{1}{2}$, then the first equation reads $\frac{3}{4} = b^2$, or equivalently, $b = \pm\frac{\sqrt{3}}{2}$. Therefore we have two more solutions of the equation $z^2 = \overline{z}$, namely $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

With these, we excluded all possibilities.

To summarize, there are four complex number solutions of the equation $z^2 = \overline{z}$. They are

$$z = 0, \quad z = 1, \quad z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$